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Advanced and Retarded Solutions in Field Theory

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Abstract

The appearance of retarded and advanced solutions to the wave equation and the *ad hoc* choice of one of them in preference to the other has led to the claim that electromagnetic theory is incomplete. It is contended that some other condition outside that theory must be invoked in order to make a meaningful physical choice. However, both a physical and mathematical analysis show, that, in fact, both retarded and advanced solutions are necessary and that the rejection of the advanced type of solution is not valid; that its rejection is based on an ambiguous mathematical technique and a faulty physical interpretation. A unique form for the Green's function is obtained which is related to all other types of Green's functions by the appropriate adjunction of solutions to the homogeneous equation.

Introduction

Solutions to Maxwell's electromagnetic field equations involve, in general, a component termed a retarded solution, another called an advanced solution, and a third element designated the free field solution. Classical electrodynamics rejects the advanced solution on the grounds that it implies that the effect precedes its cause. Consequently, the retarded solutions and those for the homogeneous equation have been, until recently, considered the only physically acceptable solutions.

The reasons for reconsidering the present situation can be found in a number of past and recent publications. It is contended in some (see, for example, Hoyle & Narlikar, 1969), that the *ad hoc* rejection of the advanced solutions is unacceptable; as being equivalent to the imposition of a condition which is not specified by the physical situation. In fact, the exclusion of the advanced solutions is tantamount to admitting that electromagnetic theory is incomplete. Others would like to include these solutions so that an action-at-a-distance formalism could be developed. Behind this desire are a number of problems which have not been satisfactorily resolved (Wheeler & Feynman, 1949).

The purpose of this report is to show that on both physical and mathematical grounds, we cannot dispense with the advanced solutions. They are essential for a description of the electromagnetic field, and their rejection is based partly on an inadequate mathematical technique and a misleading physical interpretation.

We will consider the physical aspects of the problem and then follow with a mathematical analysis.

Physical Analysis

The basic objection to the use of advanced solutions to the electromagnetic equations is that they violate the notion of causality; they imply that the effect precedes the cause. To examine this argument in detail, let us consider the wave equation which is such an integral part of the theory. In an obvious notation, we have

$$\Box\psi(\mathbf{r}, t) = g(\mathbf{r}, t) \tag{1}$$

with its advanced and retarded solutions given by

$$\psi_{\pm}(\mathbf{r}, t) = \left(\frac{1}{4\pi}\right)\int d\tau' \, g\frac{(\mathbf{r}', t \pm R)}{R}; \qquad (c = 1) \tag{2}$$

$\psi(\mathbf{r}, t)$ represents the field variable, be it the electric or magnetic field, or the vector or scalar potential, or a similar quantity. The source of the field is given by $g(\mathbf{r}, t)$. R represents the distance between the field point, $\mathbf{r}$, and that portion of the source located at $\mathbf{r}'$. The plus subscript on the field variable designates the advanced solution that is given by the function in the integrand which contains the 'advanced' time $t + R$. The minus subscript denotes the retarded solution. We have set the velocity of propagation, c, equal to 1.

The retarded solution is considered the preferred representation for the physical situation because it offers the following description of an event: the observer at the point, $(\mathbf{r}, t)$, experiences a field which is a consequence of the effects of sources which appeared previously. On the other hand, the advanced solution describes the field observed at $(\mathbf{r}, t)$ as the resultant of the effects of sources which will appear later. In other words, the future is determining the present; on the face of it, a description that is physically unacceptable.

However, let us examine the wave equation which presents us with both these solutions. The source term, $g(\mathbf{r}, t)$, is specified for all space-time; it is given for the future as well as the past. Within such a framework, there is every reason to expect a specified future distribution to determine the present as meaningfully as a given past distribution. We would never be able to uncover the past were it not possible to have a future distribution determine the present. After all, the present is the future of the past.

A much stronger case can be made for the need for the advanced solution. It rests on measurement theory and, in essence, is the same argument that Einstein presents for the definition of simultaneity. For, in that definition, the time to be associated with a field point is defined to be the arithmetic average of the retarded and advanced times as measured at the source.

In other words, the present, the now of an event at some field point, is defined in terms of a retarded time—the time a light signal leaves the observer—and an advanced time—the time when the light signal returns from the field point to its source. Therefore, there is no reason to reject, out of hand, advanced solutions if such appear. In general, we should expect both types of solutions to appear.

We will find our expectations realized when we explore the mathematical aspects of the problem.

Mathematical Analysis

As was noted above, there are three types of solution, the retarded, the advanced and the homogeneous. Mathematical theory informs us that the difference between the retarded and advanced solution must be a solution to the homogeneous equation. Thus, one is tempted to say that the advanced solution is a linear combination of the retarded and homogeneous solutions. Therefore, it is not at all clear that the retarded or advanced type of solution usually referred to is free of the other. This difficulty has not received the attention it requires. The usual mathematical procedures and techniques which are used to establish the retarded and advanced solutions do not assure us that the final results are free of solutions to the homogeneous equation. If we could eliminate such contributions, then we would obtain the irreducible contribution to the overall solution by the retarded and advanced solutions. A procedure to achieve this goal will now be developed.

The essence of the method is to so define the domain of the D'Alembertian operator, $\Box \equiv \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$, that its inverse, $\Box^{-1}$, exists. Then it follows that from the wave equation,

$$\Box f = g,$$

we obtain the result,

$$f = \Box^{-1} g$$

Hence, if $g = 0$, then so is f.

In other words, a properly restricted D'Alembertian operator will not have solutions to the homogeneous equation. The construction of $\Box^{-1}$ insures that the solution we obtain will depend only on the source, and will be free of any contribution from solutions to the homogeneous equation. Once such a solution has been determined, we can add the requisite solutions to the homogeneous equation so that conditions on the general solution can be satisfied.

To obtain the particular solution, we proceed by determining the appropriate Green's function for the D'Alembertian operator. Under the proposed restriction, that the inverse operator, $\Box^{-1}$, exists, there is but one such function. This fact alone sets the present development apart from the conventional procedure. The latter produces a variety of Green's functions which differ from each other by solutions to the homogeneous equation. It follows that a conventionally determined Green's function must be equal to the unique Green's function, which we will derive, plus solutions to the homogeneous equation.

The derivation of the unique Green's function proceeds in two steps. The first of these is the use of Fourier transforms over the space variables and a Laplace transform over the time variable. Thus, starting from the equation,

$$\Box G(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') \tag{3}$$

and proceeding along well-known lines, we obtain,

$$(k_0{}^2 + k^2)\, \hat{G}(\mathbf{k}, k_0; \mathbf{r}', t') = (2\pi)^{-3/2} H(t') \exp(-k_0 t' + ik_l x_l) \tag{4}$$

where $k^2 = k_1{}^2 + k_2{}^2 + k_3{}^2$; $k_l x^l$ is summed over l for $l = 1, 2, 3$, and the Heaviside step function is defined by

$$\begin{aligned} H(t) &= 1; \qquad t > 0 \\ &= 0; \qquad t < 0 \end{aligned}$$

and

$$\hat{G}(\mathbf{k}, k_0; \mathbf{r}', t') = (2\pi)^{-3/2} \int_0^\infty dt \exp(-k_0 t) \iiint_{-\infty}^{\infty} d^3x \exp(ik_l x^l)\, G(\mathbf{r}, t; \mathbf{r}', t')$$

Had this problem been solved by using the conventional approach, i.e., taking the Fourier transform over space and time, we would have obtained the factor $k_0{}^2 - k^2$ instead of the factor $k_0{}^2 + k^2$ which appears in equation (4). The former introduces singularities which result in the derivation of a number of Green's functions that arise from the manner in which the singularities are circumvented. However, the factor $k_0{}^2 + k^2$ presents no difficulties, and we determine a unique Green's function. Its limitation is that it only spans the positive time axis, but this restriction is easily remedied.

Using the appropriate inversion theorems for the respective transforms, we readily calculate that

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{H(t')}{4\pi R}\, \delta(t - t' - R) \tag{5}$$

with $R = |\mathbf{r} - \mathbf{r}'|$.

Notice that for this positive time interval we obtain the retarded form for the solution.

The second step is a repetition of the first, except that the time interval is $t < 0$. For this case, we again use a Fourier transform over the space variables, but an altered Laplace transform over the time variable, whose range is now $-\infty < t0$. Such a transform, $\hat{f}(p)$, for a function $f(t)$ is defined by

$$\hat{f}(p) = \int_{-\infty}^{0} \exp(pt)\, dt f(t); \qquad \mathrm{Re}\, p > 0$$

and its inverse is given by

$$f(t) = \left(\frac{1}{2\pi i}\right) \int_{c-i\infty}^{c+i\infty} \exp(-px)\, dp\hat{f}(p); \qquad c > 0$$

Carrying through the operations in the usual manner, we find that

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{H(-t')}{4\pi R}\,\delta(t' - t - R) \tag{6}$$

Thus, for $t < 0$, the Green's function is of the advanced type.

Combining both results, we find the Green's function that is applicable over the entire range for t,

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{4\pi R}\{H(t')\,\delta(t - t' - R) + H(-t')\,\delta(t' - t - R)\} \tag{7}$$

If we apply the result to the solution of equation (1), we find

$$\psi(\mathbf{r}, t) = \int \frac{d^3x'}{4\pi R}\,[H(t - R)\,g(\mathbf{r}', t - R) + H(-t - R)\,g(\mathbf{r}', t + R)] \tag{8}$$

Both equations (7) and (8) indicate the essential role of the retarded and advanced components in the solution to the wave equation. For $t > 0$, the retarded component in equation (8) determines the field quantity $\psi(\mathbf{r}, t)$, whereas for $t < 0$ the advanced component determines the field quantity. In physical terms, the retarded component accounts for the future, whereas the advanced component determines the past history of the field quantity.

Another feature of this solution, which is absent from the conventional solution, is the restriction on the integration over the spatial distribution of the source: equation (8) requires that $|t| \geqslant R$, while the solutions given by equation (2) are not so restricted. Herein lies the heart of the matter.

Discussion

Equations (2) and (8) are, undoubtedly, solutions to the wave equation (1). What we have shown above is that equation (8) is that particular solution of the wave equation which depends only on the source distribution and is free of solutions to the homogeneous equation. The latter type of solution must be introduced so that boundary conditions can be satisfied. However, equations (2), by their very derivation, require the satisfaction of boundary conditions in order that the appropriate treatment of the singularities which appear can be applied. Therefore, these solutions contain contributions from the homogeneous solutions to the wave equation, and it is not obvious that the advanced and retarded solutions are intrinsically two distinct solutions for the same physical situation. In fact, as we have pointed out, the difference between these 'distinct' solutions is a solution to the homogeneous equation. Therefore, it is not an *ad hoc* assumption to choose one of the two 'distinct' functions as representative of a given physical event. The particular choice made is immaterial, so that the preference for the retarded solution, although given plausible physical grounds for its choice, can be said to be *ad hoc.*

We can best illustrate these remarks by expressing the Green's functions associated with the advanced and retarded solutions in terms of the unique Green's function derived above. For the advanced Green's function, we have

$$\begin{aligned}
G(\text{advanced}) &= \frac{1}{4\pi R}\delta(t'-t-R) \\
&= \frac{1}{4\pi R}\{H(t')\,\delta(t'-t-R)+H(-t')\,\delta(t'-t-R)\} \\
&= \frac{1}{4\pi R}\{H(t')\,\delta(t-t'-R)+H(-t')\,\delta(t'-t-R)\} \\
&\quad + \frac{H(t')}{4\pi R}\{\delta(t'-t-R)-\delta(t-t'-R)\}
\end{aligned}$$

We recognize the first term on the right as the unique Green's function of this paper plus a term which is a non-singular solution to the homogeneous wave equation. A similar rearrangement for the retarded Green's function yields the relation

$$G(\text{retarded}) = G(\text{unique}) + \frac{H(-t')}{4\pi R}\{\delta(t-t'-R)-\delta(t'-t-R)\}$$

These equations exhibit in full the relationships among the various Green's functions, and clearly demonstrate that the solution to the electromagnetic equations is unique. The 'device' of the retarded solution is illusory. Moreover, an arbitrary combination of advanced and retarded solutions may be chosen but the conditions of the physical event being described will determine a unique solution. The intrinsic role for the retarded and advanced solutions appears in the unique Green's function which was derived above; all other formulations merely incorporate the additional homogeneous solutions needed to satisfy boundary conditions.

Conclusion

We have shown that there is a unique solution to the wave equation associated with the determination of a unique Green's function. The role of retarded and advanced effects is clearly delineated and there is no 'choice' between retarded and advanced solutions. This result denies the thesis, recently raised, that electromagnetic theory is incomplete. Moreover, the use of combinations of advanced and retarded solutions as *independent* aspects of the electromagnetic field is misleading. The 'advanced' and 'retarded' solutions are not mutually exclusive. In fact, we have shown that both are expressible in terms of a unique Green's function plus solutions to the homogeneous equations.

It is not to be inferred that there is no significance to the concepts of retarded and advanced solutions, but these are, when interpreted in accordance with their appearance in the unique Green's function, in harmony with our physical 'intuition' or experience. Thus, equation (7) reads that for $t > 0$, the field variable will be determined by the retarded portion of the Green's function—just the physical causality relation we expect. Whereas for $t < 0$ the field variable will be determined by the advanced segment of the Green's function. Physically, this means that past values of the field variable are determined by the present (which is the future or advanced aspect of the past). A not surprising result in a classical deterministic theory.

References

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